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APPLICATION OF THE METHODS OF ANALYTICAL DYNAMICS TO THE THEORY OF OPTIMAL FLIGHTS

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The author gives an exposition of the theory of optimal processes, based on the methods of analytical dynamics. By integrating the variational formula of the action functional over the trajectories or arcs of a special field of extremals, the maximum principle is proved, and the sufficient conditions of the extremum are established. The possible application of the Hamilton-Jacobi method to the construction of optimal regimes is pointed out; starting from that method, equations defining the optimal impulse transfers between coplanar orbits are obtained.

Author

Section 1. The Conditions of Stationarity

Given: a controllable device whose motion is described by the system of ordinary differential equations:

$$\dot{x}_i = F_i(x_j, u_k, t) \quad (i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, r). \tag{1.1}$$

where the $u_k(t)$ are the controls or the control functions, which may be piecewise continuous; and t is the time.

The controls must satisfy the conditions:

$$u_{k1} \leqslant u_k \leqslant u_{k2},\ u_{k1} = \text{const},\ u_{k2} = \text{const}. \tag{1.2}$$

In particular, for all or some equations, the equalities $u_{k1} = -\infty$, $u_{k2} = \infty$ may be satisfied.

* Numbers in the margin indicate pagination in the original foreign text.

We shall assume that to any admissible equation there corresponds a unique arc system (1.1), starting from the point with the coordinates x_i^0 at the initial time of motion t_0. The end-time will be denoted by T, and the coordinates of the end point by x_i^T.

The boundary values of the variables, in the general case, are correlated by the implicit conditions

$$R_\rho(x_i^0,\ x_i^T,\ t_0,\ T)=0 \qquad (\rho=1,\ 2,\ \ldots,\ l\leqslant 2n+2). \tag{1.3}$$

We shall call a trajectory allowable or admissible if it satisfies the relations (1.1) - (1.3).

Required: to select the controls in such a manner that the functional

$$J=\int_{t_0}^{T} F_0(x_i,\ u_k,\ t)\,dt+R_0(x_i^0,\ x_i^T,\ t_0,\ T), \tag{1.4}$$

calculated for these controls shall be minimal as compared with the values /134
of the functional for all other admissible arcs.

The functional J on the admissible arcs is equivalent to the functional

$$U=\int_{t_0}^{T} L\,dt+R, \tag{1.4'}$$

where

$$L=F_0+\sum_{i=1}^{n}\lambda_i(\dot{x}_i-F_i), \qquad R=R_0+\sum_{\rho=1}^{l}\nu_\rho R_\rho.$$

Here, λ_i are the explicit functions of time to be determined and ν_ρ are indeterminate constants.

From the controls u_k we may pass to the new controls γ_k by means of the formulas $u_k = u_k(\gamma_k)$, whose derivatives $\frac{du_k}{d\gamma_k}$ vanish on the boundaries of the region (1.2). Examples of the introduction of such controls are discussed in several papers (Bibl.1.2).

Instead of expressing u_k in terms of γ_k, a larger number of equations can be considered and auxiliary indeterminate multipliers λ_{n+k}, which are functions of time, can be introduced. To the function L, in that case, we must add the sum

$$\sum_{k=1}^{r}\lambda_{n+k}\left[u_k-u_k(\gamma_k)\right].$$

The total variation of the action functional will be defined by the formula

$$\Delta\int_{t_0}^{T}L\,dt=\int_{t_0}^{T}\left[\sum_{k=1}^{r}\frac{\partial L}{\partial\gamma_k}\delta\gamma_k-\sum_{i=1}^{n}\left(\dot{\lambda}_i+\frac{\partial H}{\partial x_i}\right)\delta x_i\right]dt+\left(\sum_{i=1}^{n}\lambda_i\Delta x_i-H\Delta t\right)_{t_0}^{T}. \tag{1.5}$$

for which the following notation has been adopted:

$$H=\sum_{i=1}^{n}\lambda_i\dot{x}_i-L=-F_0+\sum_{i=1}^{n}\lambda_i F_i. \tag{1.6}$$

If the controls γ_k are not varied, then eq.(1.5) coincides with the formula of the total variation of the action functional, which is extensively used in analytical mechanics (Bibl.3).

Let the functional J have a smooth maximum. Then, by equating ΔU to zero, we arrive at the necessary conditions of the extremum

$$\dot{\lambda}_i=-\frac{\partial H}{\partial x_i}\qquad (i=1,\ 2,\ \ldots,\ n), \tag{1.7}$$

$$\frac{\partial H}{\partial\gamma_k}=0\qquad (k=1,\ 2,\ \ldots,\ r), \tag{1.8}$$

$$\Delta R+\left(\sum_{i=1}^{n}\lambda_i\,\Delta x_i-H\Delta t\right)_{t_0}^{T}=0. \tag{1.9}$$

By expressing, in eq.(1.9), the total variations of the variables x_i in terms of isochronous variations by the formulas $\Delta x_i = \delta x_i + \dot{x}_i\Delta t$, and equating to zero the terms in Δt and δx_i at the times t_0 and T, the conditions (1.9) will be reduced to the form of

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$$\left.\begin{aligned}&-L_0+\frac{\partial R}{\partial t_0}+\sum_{i=1}^{n}\frac{\partial R}{\partial x_i^0}\dot{x}_i(t_0)=0,\\&L_T+\frac{\partial R}{\partial T}+\sum_{i=1}^{n}\frac{\partial R}{\partial x_i^T}\dot{x}_i(T)=0,\\&-\lambda_i^0+\frac{\partial R}{\partial x_i^0}=0,\ \lambda_i^T+\frac{\partial R}{\partial x_i^T}=0.\end{aligned}\right\} \tag{1.10}$$

The conditions (1.7), (1.8), and (1.10) are necessary conditions for the variational problem of Bolza (Bibl.4). If t_0 is prescribed, the first equation of the system (1.10) must be omitted and if T is prescribed, the second equation of that system.

We now note the important problem when the functional to be minimized need not have a smooth minimum. This is a rapid-action problem. In this case, $F_0 = 1$, $R_0 = 0$, $\Delta\int_{t_0}^{T} L\,dt=\Delta T-\Delta t_0$, so that eq.(1.5) takes the form

$$\int_{t_0}^{T}\left[\sum_{k=1}^{r}\frac{\partial L}{\partial\gamma_k}\delta\gamma_k-\sum_{i=1}^{n}\left(\dot{\lambda}_i+\frac{\partial H}{\partial x_i}\right)\delta x_i\right]dt+\left[\sum_{i=1}^{n}\lambda_i\Delta x_i-(1+H)\,\Delta t\right]_{t_0}^{T}=0.$$

Hence we obtain the equality $H_0 = H_T = -1$, as well as eqs.(1.7), (1.8) and the conditions of transversality

$$\Delta R+\left(\sum_{i=1}^{n}\lambda_i\,\Delta x_i\right)_{t_0}^{T}=0. \qquad (1.9')$$

The function R, in this case, is defined by the formula

$$R=\sum_{\rho=1}^{l}\nu_\rho R_\rho .$$

The necessary conditions for other optimal problems with a non-smooth extremum of the fundamental functional are obtained similarly.

If a larger number of controls are being considered, then eqs.(1.8) are replaced by the relations

$$\frac{\partial H}{\partial u_k}-\lambda_{n+k}=0,\ \ \lambda_{n+k}\frac{du}{d\gamma_k}=0. \qquad (1.8')$$

Now let the controls γ_k not be introduced, and let us investigate the problem with only the original controls u_k. In this case, eqs.(1.8) must be replaced by the conditions

$$\frac{\partial H}{\partial u_k}=0 \ \text{ at } \ u_{k1}<u_k<u_{k2},$$
$$\frac{\partial H}{\partial u_k}\leqslant 0 \ \text{ at } \ u_k=u_{k1},\ \ \frac{\partial H}{\partial u_k}\geqslant 0 \ \text{ at } \ u_k=u_{k2}. \qquad (1.8'')$$

<u>Note 1</u>. The system of equations (1.1) and (1.7) is a canonical system with the Hamiltonian H, the pulses λ_i, and the coordinates x_i. This canonical system differs from the usual canonical system in that H depends linearly on λ_i, and in that the additional variables u_k and γ_k are present.

Since the system of equations (1.1) and (1.7) is canonical, while the boundary controls u_{k1} and u_{k2} are constant, the total time derivative of the /136 function H will be equal to the partial time derivative of this function with the group of variables x_i, λ_i, u_k, t. Hence, on any phase of the optimal trajectory with continuous controls, we shall have:

$$H = h + \int_{t_0}^{t} \left(-\frac{\partial F_0}{\partial t} + \sum_{i=1}^{n} \lambda_i \frac{\partial F_i}{\partial t} \right) dt, \tag{1.11}$$

where h = const and t_0 is the initial time of that phase.

If the functions F_0 and F_i do not explicitly depend on t, then H will be constant on any such portion.

<u>Note 2</u>. A point at which the controls undergo a discontinuity is called a corner. Let us vary the position and time of some corner point by the quantities Δx_i and $\Delta\tau$. Then, from eq.(1.5), the variation of the fundamental functional U, taken on the optimal arc before and after the corner, will be

$$\left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t \right)_{\tau-0} - \left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H \Delta t \right)_{\tau+0},$$

where $$\Delta x_i |_{\tau-0} = \Delta x_i |_{\tau+0} = \Delta x_i, \ \Delta t |_{\tau-0} = \Delta t |_{\tau+0} = \Delta\tau.$$

This variation of the functional must vanish for the optimal arc. Hence we obtain the well-known Weierstrass-Erdmann conditions of continuity of λ_i and H along the entire optimal arc.

For this reason, when the functions F_0 and F_i are stationary, the Hamiltonian H will be one and the same constant for the entire optimal arc. For example, for the rapid-action problem, as follows from the above discussion, it will be equal to minus unity.

Section 2. The Maximum Principle

The method of integrating the variational formula of the action functional over the tubes of the trajectories is widely used in analytical mechanics (see Bibl.4). Let us apply it to find the necessary and sufficient conditions of the extremum.

Consider any admissible arc AB of the system (1.1) and (1.7) with the controls $\{\tilde{u}_k\}$. This arc is made to pass through any two points A and B satisfying eqs.(1.3) at certain times $\tilde{t}_0$ and $\tilde{T}$. In particular, the points A and B, as well as the times $\tilde{t}_0$ and $\tilde{T}$, may coincide with the initial and final positions and times of the optimal arc, and the arc AB can differ from the optimal only over a certain interval for which the controls $\{\tilde{u}_k\}$ differ from the controls $\{u_k\}$ of the optimal arc.

Let us now pass through the point A and the initial point of the optimal arc in the (2n + 1)-dimensional space of configurations of the coordinates x_i, impulses λ_i and time t, a certain curve C_u^0:

$$x_i^0 = x_i^0(\alpha_p), \quad \lambda_i^0 = \lambda_i^0(\alpha_p), \quad t^0 = t(\alpha_p, v^0), \quad \alpha_p = \alpha_p(\mu). \quad (p = 1, 2, \ldots, 2n). \tag{2.1}$$

We note that for the time being eqs.(2.1) may be arbitrarily assigned, subject to the requirements that $\frac{\partial t}{\partial v} > 0$, and that the contour of C_u^0 passes through the above two points. The additional restrictions on the choice of C_u^0 will be given below.

We shall apply the term extremal to any arc of the system (1.1) and /137 (1.7). Let us now pass through points of the curve AB and that part of the curve C_u^0 included between the initial point of the optimal curve and the point

A, the extremals corresponding to controls equal to the controls $\{u_k\}$ on the optimal arc. Let us continue the contour G_μ^0 beyond the point A so that the extremals crossing the curve AB shall intersect it.

Let the relations (2.1) be such that for $v = v^T$ the curve G_μ^0 shall be transformed by these extremals into the curve G_μ passing through the end point of the optimal arc and the point B. The contour of G_μ will be defined by the general formulas

$$\left.\begin{aligned} x_i &= x_i(x_j^0(\alpha_p),\ u_k(t),\ t), \\ \lambda_i &= \lambda_i(\lambda_j^0(\alpha_p),\ x_j^0(\alpha_p),\ u_k(t),\ t), \\ t &= t(\alpha_p,\ v^T),\ \alpha_p = \alpha_p(\mu). \end{aligned}\right\} \tag{2.2}$$

Through the points of this segment of the curve G_μ^0 let us draw extremals with controls $\{u_k'\}$ such that these extremals, for $v = v^T$, shall completely fill that segment of the curve G_μ between the end point of the optimal arc and the point B, if the following conditions are satisfied:

$$u_k'(t(\alpha_p(\mu),\ v^0)) = u_k(t(\alpha_p(\mu),\ v^0)),$$
$$u_k'(t(\alpha_p(\mu),\ v^T)) = u_k(t(\alpha_p(\mu),\ v^T)).$$

At any time t the boundary values for the controls $\{u_k'\}$ will be $\{u_k\}$ and $\{\tilde{u}_k\}$.

Let us now subject the choice of the curve G_μ^0 and the controls $\{u_k'\}$ to the final condition:

$$\Delta R + \left(\sum_{i=1}^{n} \lambda_i \Delta x_i - H\,\Delta t\right)_{t(\alpha_p,\,v^0)}^{t(\alpha_p,\,v^T)} = 0. \tag{2.3}$$

Let us write out the condition (2.3) in the expanded form:

$$\sum_{p=1}^{2n}\left[\frac{\partial}{\partial\alpha_p} R_0(x_i^0,\ x_i^T,\ t_0,\ t^T) + \sum_{i=1}^{n} \lambda_i^T \frac{\partial x_i^T}{\partial\alpha_p} - H_T \frac{\partial t^T}{\partial\alpha_p} - \right.$$
$$\left. - \sum_{i=1}^{n} \lambda_i^0 \frac{\partial x_i^0}{\partial\alpha_p} + H_0 \frac{\partial t^0}{\partial\alpha_p}\right] \frac{d\alpha_p}{d\mu} = 0. \tag{2.4}$$

The following notation has been used in eq.(2.4):

$$
\left.
\begin{aligned}
&x_i^0 = x_i^0(\alpha_p(\mu)),\ \lambda_i^0 = \lambda_i^0(\alpha_p(\mu)),\ t^0 = t(\alpha_p(\mu),\ v^0),\\
&x_i^T = x_i(x_j^0(\alpha_p(\mu)),\ u_k(t(\alpha_p(\mu),\ v^T)),\ t(\alpha_p(\mu),\ v^T)),\\
&\lambda_i^T = \lambda_i(\lambda_j^0(\alpha_p(\mu)),\ x_j^0(\alpha_p(\mu)),\ u_k(t(\alpha_p(\mu),\ v^T)),\ t(\alpha_p(\mu),\ v^T)),\\
&t^T = t(\alpha_p(\mu),\ v^T),\\
&H_0 = -F_0(x_j^0,\ u_k(t^0),\ t^0) + \sum_{i=1}^{n} \lambda_i^0 F_i(x_j^0,\ u_k(t_0),\ t_0),\\
&H_T = -F_0(x_j^T,\ u_k(t^T),\ t^T) + \sum_{i=1}^{n} \lambda_i^T F_i(x_j^T,\ u_k(t^T),\ t^T).
\end{aligned}
\right\}
\qquad (2.5)
$$

All the above conditions can be satisfied, because of the fact that the /138 functional dependence of x_i^0 and λ_i^0 on α_p is arbitrary, the functions α_p of the parameter μ and u_k^T of the time t are arbitrary, and the function $t(\alpha_p, v)$ is likewise arbitrary.

We shall now integrate the expressions (1.5) in the above-mentioned field of extremals with controls coinciding with the optimal controls $\{u_k\}$. We obtain

$$
-\int_{t_0}^{T} L\,dt = -\ {}_{C_\mu^0}\!\int_{t_0}^{\tilde{t}_0} \sum_{i=1}^{n} \lambda_i \Delta x_i - H\Delta t + {}_{C_\mu}\!\int_{T}^{\tilde{T}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H\Delta t - {}_{AB}\!\int_{\tilde{t}_0}^{\tilde{T}} \left(\sum_{i=1}^{n} \lambda_i \eta_i - H\right) dt. \qquad (2.6)
$$

Her[illegible] $\eta_i = F_i(x_j,\ \tilde{u}_k,\ t)$. [illegible]t side is calculated on the optimal arc, the first integral of the right side over the curve C_μ^0 from the initial point of the optimal arc to the point A; the second integral, over the curve C_μ from the end point of the optimal arc to the point B; and the third integral, over the trajectory AB.

The quantities η_i are defined by eqs.(1.1) in the form of

The function H and the factors λ_i in the third integral of the right side are calculated on the extremals of the field under consideration and the points of their intersection with the curve AB. The controls entering explicitly into H therefore coincide with the optimal controls $\{u_k\}$.

On integrating eq.(2.3) in the field of extremals with controls $\{u_k^{\dagger}\}$, we obtain

$$\int_{C_{\mu}T}^{\tilde{T}} \sum_{i=1}^{n} \lambda_i \Delta x_i - H\Delta t - \int_{C^0_{\mu} t_0}^{\tilde{t}_0} \sum_{i=1}^{n} \lambda_i \Delta x_i - H\Delta t = $$
$$= R_0(x_i^0,\ x_i^T,\ t_0,\ T) - R_0(\tilde{x}_i^{\tilde{t}_0},\ \tilde{x}_i^{\tilde{T}},\ \tilde{t}_0,\ \tilde{T}). \tag{2.7}$$

By definition of the function H [eq.(1.6)], we find

$$\sum_{i=0}^{n} \lambda_i \eta_i = H(x_i,\ \lambda_i,\ \tilde{u}_k,\ t) + L(x_i,\ \lambda_i,\ \tilde{u}_k,\ t). \tag{2.8}$$

Starting out from eqs.(1.4'), (2.6) - (2.8), and remembering that the optimal arc and the arc AB both satisfy the conditions (1.3), we have:

$$[\text{illegible}] = \int_{t_0}^{\tilde{T}} [-H(x_i,\ \lambda_i,\ \tilde{u}_k,\ t) + H(x_i,\ [\text{illegible}] \geqslant 0. \tag{2.9}$$

Hence we arrive at the necessary condition for the fundamental functional to be minimum. This condition is that H shall be maximum on the optimal arc:

$$H(x_i,\ \lambda_i,\ u_k,\ t) \geqslant H(x_i,\ \lambda_i,\ \tilde{u}_k,\ t). \tag{2.10}$$

The condition (2.10) is the fundamental condition of L.S.Pontryagin's maximum principle (Bibl.5) and is equivalent to the necessary Weierstrass condition (Bibl.4). More particularly, the conditions (1.8), (1.8') and (1.8") follow from the relation (2.10). /139

Let condition (2.10) be satisfied for any controls $\{\tilde{u}_k\}$, some controls $\{u_k\}$, and for x_i and λ_i which are a solution of eqs.(1.1), (1.7) for any x_i^0, λ_i^0 and for these controls $\{u_k\}$. If, in this case, on some arc with the controls $\{u_k\}$, the conditions (1.1) - (1.3), (1.7), and (1.9) are satisfied, then, as shown by the above discussion, that arc will be optimal.

We have reached the conclusion that the maximality of H for certain controls $\{u_k\}$ with x_i and λ_i that are a solution of the system (1.1) and (1.7), for arbitrary initial values, and the satisfaction of the conditions (1.1) - (1.3), (1.7), and (1.9) for a trajectory with these controls, are sufficient conditions for the optimality of these controls.

Note. The necessity of condition (2.10) for an optimal arc can be shown by means of the following simple construction:

Consider the field of extremals corresponding to controls equal to the optimal controls $\{u_k\}$ and to the arc AB, differing from the optimal only over a certain internal interval which can be as small as may be desired. Let this interval include the point (x_i, λ_i, t) for which condition (2.10) is to be proved. Let us draw the arc AB outside the optimal arc such that it can be either tangent to certain extremals or intersect them, but necessarily at two points.

On integrating eq.(1.5) over the extremals intersecting the arc AB, we obtain, instead of eq.(2.6),

$$-\int_{t_0}^{T} L\,dt = -{}_{AB}\int_{t_0}^{T}\left(\sum_{i=1}^{n}\lambda_i\eta_i - H\right)dt. \qquad (2.6')$$

By means of eqs.(2.6') and (2.8) we arrive at eq.(2.9), and consequently also at the necessary condition (2.10).

Section 3. Application of the Hamilton-Jacobi Method

The optimal trajectory $\{x_i\}$ and the indeterminate factors $\{\lambda_i\}$ satisfy the canonical equation with the Hamiltonian function (1.6) with controls $\{u_k\}$ which are to be considered as explicit functions of time. This canonical system may be solved by the Hamilton-Jacobi method.

For this purpose, let us set up the partial differential equation

$$\frac{\partial W}{\partial t}+H\left(x_i, \frac{\partial W}{\partial x_i}, u_k(t), t\right)=0. \tag{3.1}$$

On the basis of eq.(1.6), eq.(3.1) will be of the form

$$\frac{\partial W}{\partial t}+\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}F_i=F_0. \tag{3.1'}$$

Equation (3.1') shows that the Hamilton-Jacobi equation for the problem of the selection of optimum controls is a first-order linear partial differential equation.

Let, for certain $u_k(t)$, the following complete integral of eq.(3.1) be found: /140

$$W=W(x_i, a_i, t)+a_0, \tag{3.2}$$

where a_i are n independent constants and a_0 is an additive constant. Then, the solution of eqs.(1.1) and (1.7) for these controls $u_k(t)$ is given by

$$\frac{\partial W}{\partial a_i}=b_i, \tag{3.3}$$

$$\frac{\partial W}{\partial x_i}=\lambda_i. \tag{3.4}$$

where b_i are new independent constants.

It is generally known that the complete integral of eq.(3.1) is determined, with an accuracy to an additive constant, by the action integral. We, therefore, have

$$W=\int_{t_0}^{t} L\,dt+a_0.$$

The function R of eq.(1.4') depends on the initial and final values of the variables and will be a certain additive constant. Hence, the functional to be minimized will be equal, with accuracy to within an additive constant, to the complete integral of eq.(3.1).

On the basis of eqs.(3.1) and (3.4), for any explicit representation of a functional on the trajectories of the system (1.1) and (1.7), we have the relations

$$\frac{\partial U}{\partial x_i} = \lambda_i, \quad \frac{\partial U}{\partial t} = -H. \tag{3.5}$$

Equations (3.5) may be used to work out programs of numerical solution of variational problems, for instance, by the divergence method (Bibl.6).

If $a_i = \lambda_i^0$ and $b_i = x_i^0$, then the complete integral of the Hamilton-Jacobi equations is written in the form of (Bibl.7)

$$W(x_i, \lambda_i^0, t) = \int_{t_0}^{t} L\,dt + \sum_{i=1}^{n} x_i^0 \lambda_i^0. \tag{3.6}$$

Section 4. Optimal Impulse Transfers between Coplanar Orbits

As an example of the application of the Hamilton-Jacobi method, let us find the conditions of optimal impulsive coplanar transfers between elliptical orbits in the gravitational field of a spherosymmetric central body.

Let us consider the motion of a rocket ship in a polar coordinate system (r, φ) with its origin at the center point of the central body. The equations of motion will be of the form

$$\left.\begin{aligned} \dot v_r &= v_\varphi^2 r^{-1} - k^2 r^{-2} \dot\beta \cos\psi, \\ \dot v_\varphi &= -v_r v_\varphi r^{-1} + \dot\beta \sin\psi, \\ \dot r &= v_r, \quad \dot\varphi = v_\varphi r^{-1}. \end{aligned}\right\} \tag{4.1}$$

where k^2 is the gravitational constant; v_r and v_φ are the projections of the velocity of the ship on the radius vector and transversal, respectively; $\beta = -u_r \ln m$; μ_r is the constant effective exhaust velocity; m is the mass; ψ is the angle between the thrust and the radius vector, measured in the direc- /141 tion from the radius vector to the velocity of motion of the ship.

For any possible interplanetary flights, the acceleration $\dot\beta$ developed by modern rocket engines considerably exceeds the other terms on the right sides of eqs.(4.1). Therefore, for the boost phase, let us take the equations of the impulsive criterion of velocity:

$$\dot{v}_r = \dot{\beta}\cos\psi,\ \dot{v}_\varphi = \dot{\beta}\sin\psi,\ \dot{r} = 0,\ \dot{\varphi} = 0. \qquad (4.1')$$

Starting out from the capabilities of rocket engineering and the allowable accelerations, a restriction on the control relative to the value of $\dot{\beta}$ must be adopted:

$$0 \leq \dot{\beta} \leq G,\ G = \text{const.} \qquad (4.2)$$

The equations of motion on the passive phases are obtained from the system (4.1) by setting $\dot{\beta} = 0$.

Understanding the term optimal in the sense of minimizing the consumption of mass, we shall minimize the functional

$$\int_{t_0}^{T} \dot{\beta} dt = u_r \ln(m_H m_K^{-1}). \qquad (4.3)$$

where m_H is the initial mass and m_K the final mass of the ship, while t_0 and T are the respective times of start and finish.

The functional (4.3) must be minimized by means of the constraints (4.1) or (4.1'), restriction of the control of $\dot{\beta}$ in the form of eq.(4.2), and the boundary conditions (Bibl.8):

$$v_r^H = k e_H p_H^{-1/2}\sin f_H,\ v_\varphi^H = k p_H^{1/2} r_H^{-1},\ r_H = p_H(1 + e_H\cos f_H)^{-1},\ \varphi_H = f_H + \omega_H, \qquad (4.4)$$

$$v_r^K = k e_K p_K^{-1/2}\sin f_K,\ v_\varphi^K = k p_K^{1/2} r_K^{-1},\ r_K = p_K(1 + e_K\cos f_K)^{-1},\ \varphi_K = f_K + \omega_K. \qquad (4.5)$$

Here the subscript H indicates the characteristics of the initial orbit and the subscript K those of the terminal orbit. The following notation is used in eqs.(4.4) and (4.5): e = eccentricity; p = parameter; ω = angular distance of the pericenter from the polar axis; f = true anomaly.

In the case under consideration, $R_0 = 0$ and the boundary conditions are expressed by the explicit relations (4.4) and (4.5). For this reason we shall not introduce the indeterminate constants ν_ρ, and shall use the conditions of transversality in the general form:

$$(\lambda_1\Delta v_r + \lambda_2\Delta v_\varphi + \lambda_3\Delta r + \lambda_4\Delta\varphi - \text{[illegible]} = 0.$$

Here the total variations of the variables are calculated along the limiting orbits. In view of eqs.(4.4) and (4.5), the variations of all variables except Δt are expressed respectively in terms of Δf_H and Δf_K. In formulating the problem we do not consider the specific motions along the limiting orbits, so that the variations of Δf_H, Δf_K, Δt_0 and Δt will be independent. Hence we obtain H_H, H_K = 0 and, since eqs.(4.1) are stationary, we obtain $H = 0$ along the entire optimal trajectory. The variations of velocities and coordinates on the limiting orbits are proportional to the right sides of eqs.(4.1), calculated on the limiting orbits for $\dot{\beta} = 0$.

We arrive at the following conditions of transversality:

$$\sum_{i=1}^{n} \lambda_i^H F_i^H = 0, \quad \sum_{i=1}^{n} \lambda_i^K F_i^K = 0. \tag{4.6}$$

where F_i^H and F_i^K are the right sides of eqs.(4.1) for the initial and final orbits, respectively.

Starting out from eqs.(4.1') let us set up the function H for the active phases: /142

$$H = \dot{\beta}\,(-1 + \lambda_1 \cos\psi + \lambda_2 \sin\psi). \tag{4.7}$$

We now write the Hamilton-Jacobi equation:

$$\frac{\partial W}{\partial t} + \dot{\beta}\left(-1 + \frac{\partial W}{\partial v_r}\cos\psi + \frac{\partial W}{\partial v_\varphi}\sin\psi\right) = 0.$$

The complete integral of this equation is

$$W = a_1 v_r + a_2 v_\varphi + \int_{t_0}^{t} \dot{\beta}(1 - a_1\cos\psi - a_2\sin\psi)\,dt + a_3 r + a_4\varphi.$$

The integrals of eqs.(3.3) and (3.4) yield

$$\left.\begin{aligned} v_r - \int_{t_0}^{t} \dot{\beta}\cos\psi = b_1,\ \ v_\varphi - \int_{t_0}^{t} \dot{\beta}\sin\psi = b_2,\ \ r = b_3,\ \ \varphi = b_4, \\ a_1 = \lambda_1,\ \ a_2 = \lambda_2,\ \ a_3 = \lambda_3,\ \ a_4 = \lambda_4. \end{aligned}\right\} \tag{4.8}$$

We recall the conditions of maximum H on the optimal trajectory. Writing out the first and second partial derivatives of the function (4.7) with respect to $\dot{\beta}$ and ψ, we come to the conclusion that the active phases can include the trajectories: (a) of the programmed thrust at $\frac{\partial H}{\partial \dot{\beta}} = \frac{\partial H}{\partial \psi} = 0$

or

$$\lambda_1 = \cos\psi, \quad \lambda_2 = \sin\psi; \tag{4.9}$$

(b) phases of maximum reactive acceleration G at [illegible] $\frac{\partial H}{\partial \psi} = 0.$ Since H = 0 and H is a linear homogeneous function of $\dot{\beta}$, the sign of inequality for the derivative $\frac{\partial H}{\partial \dot{\beta}}$ is dropped and the indeterminate multipliers here are also determined from eqs.(4.9).

Equations (4.8) and (4.9) show that the inclination angle ψ of the thrust remains constant for any powered section of the trajectory. For each such phase or section, eqs.(4.1') and (4.8) will yield

$$\begin{gathered} v_r - v_r^0 = V\cos\psi, \quad v_\varphi - v_\varphi^0 = V\sin\psi, \quad r = r^0, \quad \varphi = \varphi^0, \\ \beta - \beta_0 = V, \quad V = [(v_r - v_r^0)^2 + (v_\varphi - v_\varphi^0)^2]^{1/2}. \end{gathered} \tag{4.10}$$

It follows more particularly from eqs.(4.10) that the consumption of mass during the active phase is determined by the required increment of velocity and does not depend on the particular law of burning.

On the passive phases, the controls over $\dot{\beta}$ and ψ are shut down. These phases admit of the maximum principle. Let us find an expression for the indeterminate factors on the passive phases by the Hamilton-Jacobi method.

The Hamilton-Jacobi equation is of the form

$$\frac{\partial W}{\partial t} - \frac{\partial W}{\partial v_r}(v_\varphi^2 r^{-1} - k^2 r^{-2}) - \frac{\partial W}{\partial v_\varphi} v_r v_\varphi r^{-1} + \frac{\partial W}{\partial r} v_r + \frac{\partial W}{\partial \varphi} v_\varphi r^{-1} = 0. \tag{4.11}$$

The complete integral is

$$W = a_1 t + a_2\varphi + \Phi(v_r, v_\varphi, r, a_1, a_2, a_3),$$

where Φ is a certain function of these arguments.

On the basis of eq.(3.4), this yields /143

[illegible]

For this reason, λ_4 = const for both passive and active phases. By virtue of the Weierstrass-Erdmann conditions, λ_4 is the same constant over the entire optimal trajectory.

For the passive phases, we have $\dot{\beta} = 0$, hence L = 0, and by eq.(3.6) the complete integral is written in the form of

$$W(x_i, \lambda_i^0, t) = \sum_{i=1}^{4} x_i^0 \lambda_i^0. \tag{4.12}$$

Starting from eq.(3.4), we obtain expressions for the indeterminate multipliers:

$$\lambda_i = \sum_{j=1}^{4} \lambda_j^0 \frac{\partial x_j^0}{\partial x_i}. \tag{4.13}$$

Let us take the solution of eqs.(4.1) on the passive phases in the following form:

$$v_r = kep^{-1/2} \sin f, \quad v_\varphi = kp^{-1/2}(1 + e\cos f), \quad r = p(1 + e\cos f)^{-1}, \quad \varphi = f + \omega. \tag{4.14}$$

where e, p, and ω are the eccentricity, parameter, and angular distance of the pericenter of the transitional elliptical orbit from the polar axis, and f is the true anomaly on this orbit.

We note that the true anomaly is expressed in terms of the time t by means of the relations (Bibl.8):

$$\operatorname{tg}\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}, \quad E - e\sin E = ka^{-3/2}(t - t_\pi), \tag{4.15}$$

where t_π is the time of transit through the pericenter and a is the semi-major axis, which for an elliptical orbit is equal to $p(1 - e^2)^{-1}$.

Let us denote by s_k (k = 1, 2, 3, 4) the Keplerian elements e, p, ω and t_π. Equation (4.12) then takes the form of

$$\lambda_i = \sum_{k=1}^{4} A_k \frac{\partial s_k}{\partial x_i}, \quad A_k = \sum_{j=1}^{4} \lambda_j^0 \frac{\partial x_j^0}{\partial s_k} = \text{const}. \tag{4.16}$$

If we put $t = t_0$ in eqs.(4.14) and (4.15), we arrive at the expressions

$$x_i^0 = x_i^0(e, p, \omega, t_0 - t) \tag{4.17}$$

To obtain eqs.(4.16), the Keplerian elements must be represented in terms of x_i by the aid of eqs.(4.14) and (4.15). Here, on the basis of eqs.(4.15), we shall put

$$t_0 = t + \gamma(e, p, f)$$

where γ is a certain function of these variables.

The quantities e, p, ω, and f will now be expressed on the basis of eqs.(4.14), explicitly in terms of x_j. We will, therefore, have the identities

$$\frac{\partial}{\partial t_0} x_i^0 (s_k) = \frac{\partial}{\partial t} x_i^0 (x_j, t). \tag{4.18}$$

On the basis of eq.(4.18) we have

$$A_4 = \sum_{i=1}^{4} \lambda_i^0 \frac{\partial x_i^0}{\partial t}$$

Equation (4.12) and the Hamilton-Jacobi equation will finally yield $A_4 = -H$./144 In this case, therefore, $A_4 = 0$, and the expressions for the indeterminate multipliers will contain only three indeterminate constants.

To determine the derivatives $\frac{\partial s_k}{\partial x_i}$, let us differentiate eq.(4.14) with respect to x_i. We then have

$$\sin f \frac{\partial r}{\partial x_i} - \frac{1}{2} e p^{-1} \sin f \frac{\partial p}{\partial x_i} + e \frac{\partial \sin f}{\partial x_i} = B_{1i},$$

$$\cos f \frac{\partial r}{\partial x_i} - \frac{1}{2} p^{-1} (1 + e\cos f) \frac{\partial p}{\partial x_i} + e \frac{\partial \cos f}{\partial x_i} = B_{2i},$$

$$\cos f \frac{\partial r}{\partial x_i} - p^{-1} (1 + e\cos f) \frac{\partial p}{\partial x_i} + e \frac{\partial \cos f}{\partial x_i} = B_{3i},$$

$$\frac{\partial f}{\partial x_i} + \frac{\partial \omega}{\partial x_i} = B_{4i}.$$

The coefficients B_{ij} vanish for $i \neq j$, $B_{11} = B_{22} = k^{-1} p^{1/2}$, $B_{33} = -p^{-1}(1 + e \cos f)^2$, $B_{44} = 1$.

Making use of the obvious equality

$$\sin f \frac{\partial \sin f}{\partial x_i} + \cos f \frac{\partial \cos f}{\partial x_i} = 0$$

and assuming that e $\neq$ 0, we obtain, after simple calculations,

$$\left.\begin{aligned}
&\frac{\partial e}{\partial v_r} = k^{-1}p^{1/2}\sin f, \quad \frac{\partial p}{\partial v_r} = 0, \quad \frac{\partial \omega}{\partial v_r} = -k^{-1}e^{-1}p^{1/2}\cos f, \\
&\frac{\partial e}{\partial v_\varphi} = k^{-1}e^{-1}p^{1/2}[(e^2-1)(1+e\cos f)^{-1} + e\cos f + 1], \\
&\qquad \frac{\partial p}{\partial v_\varphi} = 2k^{-1}p^{3/2}(1+e\cos f)^{-1}, \\
&\frac{\partial \omega}{\partial v_\varphi} = k^{-1}e^{-1}p^{1/2}\sin f(2+e\cos f), \quad \frac{\partial e}{\partial r} = p^{-1}(e+\cos f)(1+e\cos f), \\
&\frac{\partial p}{\partial r} = 2(1+e\cos f), \quad \frac{\partial \omega}{\partial r} = e^{-1}p^{-1}\sin f(1+\cos f), \\
&\qquad \frac{\partial e}{\partial \varphi} = \frac{\partial p}{\partial \varphi} = 0, \quad \frac{\partial \omega}{\partial \varphi} = 1.
\end{aligned}\right\} \tag{4.19}$$

Equations (4.19) are of great independent significance, since they determine the variation of the elements e, p, ω for a small variation of the position of the point and of the velocity of the satellite going into orbit.

Introducing the following notation for new constants:

$$A = -k^{-1}e^{-1}p^{1/2}A_3, \; B = k^{-1}e^{-1}p^{1/2}A_1, \; D = k^{-1}e^{-1}p^{1/2}(e^2-1)A_1 + 2k^{-1}p^{3/2}A_2,$$

we shall have, from eqs.(4.16) and (4.19):

$$\left.\begin{aligned}
&\lambda_1 = A\cos f + Be\sin f, \\
&\lambda_2 = -A\sin f(2+e\cos f) + B(1+e\cos f) + D(1+e\cos f)^{-1}, \\
&\lambda_3 = -kp^{-3/2}(1+e\cos f)^2[A\sin f(1+e\cos f)^{-1} - \\
&\qquad\qquad - B - D(1+e\cos f)^{-1}], \\
&\lambda_4 = -kep^{-1/2}A.
\end{aligned}\right\} \tag{4.20}$$

The constants A, B and D coincide with the Lawden constants (Bibl.9) for a noncircular transfer orbit. Our discussion has now established the connection of the Lawden constants with λ_i^0 and the elements of the transfer orbit. Since our study was in a polar coordinate system, eqs.(4.20) turned out simpler /145 than Lawden's corresponding formulas (Bibl.9) for the indeterminate multipliers on a noncircular orbit.

Let us obtain expressions for the indeterminate multipliers that will be suitable for any transfer orbit, including a circular one, for which e = 0.

Substituting for e and ω the new elements $q = e \cos \omega$ and $\ell = e \sin \omega$, we now write eqs.(4.14) in the form of

$$\left.\begin{array}{c} v_r = kp^{-1/2}(q \sin \varphi - l \cos \varphi), \quad v_\varphi = kp^{-1/2}(1 + q \cos \varphi + l \sin \varphi), \\ r = p(1 + q \cos \varphi + l \sin \varphi)^{-1}. \end{array}\right| \tag{4.21}$$

Let us take p, q, ℓ, and t_π as the elements s_k in eq.(4.6).

Differentiating eqs.(4.21) with respect to x_i, we obtain equations for determining the partial derivatives:

$$\left.\begin{array}{r} -\frac{1}{2} p^{-1}(q \sin \varphi - l \cos \varphi) \frac{\partial p}{\partial x_i} + \sin \varphi \frac{\partial q}{\partial x_i} - \cos \varphi \frac{\partial l}{\partial x_i} = C_{1i}, \\ -\frac{1}{2} p^{-1}(1 + q \cos \varphi + l \sin \varphi) \frac{\partial p}{\partial x_i} + \cos \varphi \frac{\partial q}{\partial x_i} + \sin \varphi \frac{\partial l}{\partial x_i} = C_{2i}, \\ -p^{-1}(1 + q \cos \varphi + l \sin \varphi) \frac{\partial p}{\partial x_i} + \cos \varphi \frac{\partial q}{\partial x_i} + \sin \varphi \frac{\partial l}{\partial x_i} = C_{3i}. \end{array}\right|$$

The matrix of coefficients $\|C_{ij}\|$ is of the form

$$\left\| \begin{array}{cccc} k^{-1}p^{1/2} & 0 & 0 & -q \cos \varphi - l \sin \varphi \\ 0 & k^{-1}p^{1/2} & 0 & q \sin \varphi - l \cos \varphi \\ 0 & 0 & -p^{-1}(1 + q \cos \varphi + l \sin \varphi)^2 & q \sin \varphi - l \cos \varphi \end{array} \right\|$$

Hence, we find the following expressions for the derivatives:

$$\left.\begin{array}{l} \frac{\partial q}{\partial v_r} = k^{-1}p^{1/2} \sin \varphi, \quad \frac{\partial l}{\partial v_r} = -k^{-1}p^{1/2} \cos \varphi, \\ \frac{\partial q}{\partial v_\varphi} = k^{-1}p^{1/2} [\cos \varphi + (\cos \varphi + q)(1 + q \cos \varphi + l \sin \varphi)^{-1}], \\ \frac{\partial l}{\partial v_\varphi} = k^{-1}p^{1/2} [\sin \varphi + (\sin \varphi + l)(1 + q \cos \varphi + l \sin \varphi)^{-1}], \\ \frac{\partial q}{\partial r} = p^{-1}(\cos \varphi + q)(1 + q \cos \varphi + l \sin \varphi), \\ \frac{\partial l}{\partial r} = p^{-1}(\sin \varphi + l)(1 + q \cos \varphi + l \sin \varphi), \\ \frac{\partial q}{\partial \varphi} = -l, \quad \frac{\partial l}{\partial \varphi} = q. \end{array}\right\} \tag{4.21'}$$

Equations (4.21) determine the variation of elements q and ℓ for a small variation in the velocity and position of the ship.

Introducing the following notation for constants:

$$M = -k^{-1}p^{1/2}A_3, \quad N = k^{-1}p^{1/2}A_2, \quad Q = 2k^{-1}p^{3/2}A_1,$$

we shall have, from eqs.(4.16) and (4.21),

$$\left.\begin{aligned}
&\lambda_1 = M\cos\varphi + N\sin\varphi,\\
&\lambda_2 = (1 + q\cos\varphi + l\sin\varphi)^{-1}\,[-M\,[2\sin\varphi + q\sin\varphi\cos\varphi +\\
&+ l(1+\sin^2\varphi))] + N\,[2\cos\varphi + q(1+\cos^2\varphi) + l\sin\varphi\cos\varphi] + Q],\\
&\lambda_3 = -kp^{-3/2}(1 + q\cos\varphi + l\sin\varphi)\,[M(\sin\varphi + l) - N(\cos\varphi + q) - Q],\\
&\lambda_4 = -kp^{-1/2}(Mq + Nl).
\end{aligned}\right\} \quad (4.22)$$

Putting $q = l = 0$ in eqs.(4.22), we obtain, for a circular transfer orbit:

$$\left.\begin{aligned}
&\lambda_1 = M\cos\varphi + N\sin\varphi,\\
&\lambda_2 = -2M\sin\varphi + 2N\cos\varphi + Q,\\
&\lambda_3 = -kp^{-3/2}(M\sin\varphi - N\cos\varphi - Q),\\
&\lambda_4 = 0.
\end{aligned}\right\} \quad (4.23)$$

The constants M, N, and Q here coincide with the Lawden constants (Bibl.9) for the case of a circular orbit. Equations (4.23) are simpler than the corresponding Lawden formulas. The difference between these also consists in the fact that our formulas, instead of the true anomaly f, use the latitute argument φ which corresponds better to a circular transfer orbit. The present study has also established the connection of the Lawden constants with λ_i^0 and the elements of a circular transfer orbit.

The general representation [eq.(4.22)] obtained above is particularly convenient for considering quasicircular transfer orbits.

The continuity of the indeterminate multipliers, their explicit expressions given by eqs.(4.9) and (4.22), the boundary conditions (4.4) - (4.5), the conditions (4.10) for velocity jumps for each active phase, and the conditions (4.6) of transversality, yield a number of equations sufficient for determining all unknown quantities for a prescribed number of impulses.

An investigation of these equations would be a separate problem, beyond the scope of this paper.

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